When all removable edges lead to wheels: (Application: characterizing graphs that are Birkhoff-von Neumann and PM-compact)

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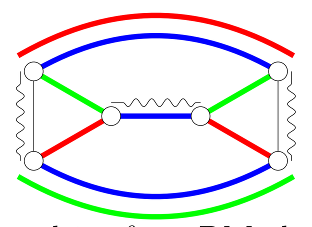
Welcome to the show

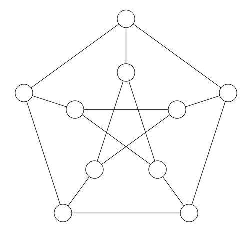
Matchings and perfect matchings have received considerable attention in graph theory as well as in other related domains (such as but not limited to algorithms and optimization). In particular, the perfect matching polytope has been studied extensively. However, there still remain many questions — pertaining to this polytope and the related graph classes – to which we do not know the answers. This research aims to obtain a simpler proof of the result by Carvalho, Kothari, Wang, Lin (2020) on the characterization of a particular graph class, that are Birkhoff-von Neumann as well as PM-compact. This proof emerges as an application of a conjecture which addresses the graphs, the removable edges of which ultimately lead to wheels.

Some mathematical jargons

1. Perfect Matching : All of the concerned graphs in our entire journey are undirected and loopless (there are no edges connecting one vertex to itself). For such a graph G, a set of edges $M \subseteq E(G)$ is called a *perfect matching* if every vertex is incident with exactly one edge in M.
2. Matching covered graph : A connected nontrivial graph G is called *matching covered* if each edge belongs to some perfect matching.
3. Removable edge : An edge e of a matching covered graph G is *removable* if $G - e$ is also matching covered.



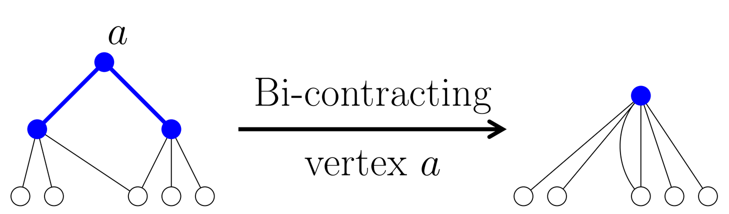
$\overline{C_6}$ and its four PM shown in rgb and squiggle



In Petersen graph $\mathbb{P}$, every edge is removable.

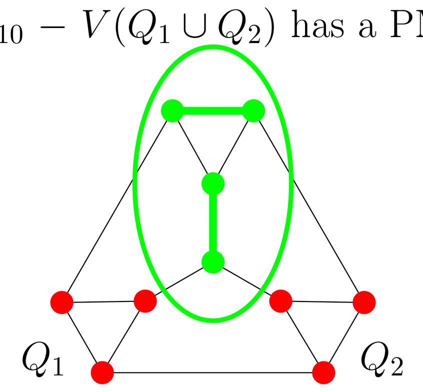
Matching covered graphs and removable edges

4. Bi-contraction : The term *bi-contraction* refers to the shrinking of a degree two vertex and its two neighbours into a single vertex. Note that if a graph G has minimum degree 3 or more, then for some edge e, $G - e$ (Let's call it H) may have at most two vertices of degree two. The bi-contraction of all these 'degree two' vertices, for such a graph H, will lead to the *retract* of H.
5. Conformal bicycle : A pair of vertex-disjoint cycles (say, Q_1 and Q_2) is called a *bicycle*. A bicycle ($Q_1 \cup Q_2$) of G is a *conformal bicycle* if $G - V(Q_1) - V(Q_2)$ has a perfect matching. The conformal bicycle is *odd* when each of Q_i ($i \in \{1, 2\}$) has odd no. of vertices and *even* if each of Q_i has even no. of vertices.
   For instance, the tricorn (R_{10}) has an odd conformal bicycle, $Q_1 \cup Q_2$ which is shown in red in the following figure. The perfect matching of $G - V(Q_1) - V(Q_2)$ is shown in green in it.



Bi contraction

$R_{10} - V(Q_1 \cup Q_2)$ has a PM.



R_{10} with $Q_1 \cup Q_2$

Bi-contraction and Conformal bicycles

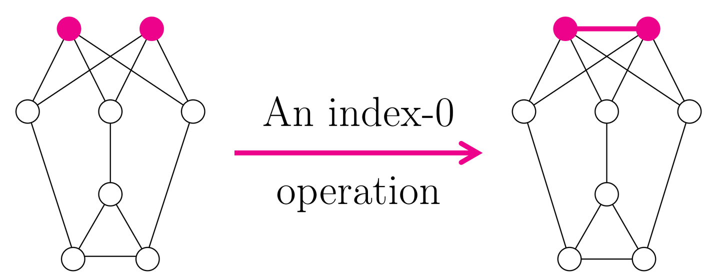
Conjecture 1

Let G be a simple matching covered graph with minimum degree 3 or more. If there exists a removable edge e in G such that retract of $G - e$ is either $K_4 \odot K_{3,3}$ or the Murty graph $\mathcal{M}$ (up to multiple spokes joining the two noncubic vertices), then

1. either G is not Birkhoff-von Neumann or G is not PM-compact, or otherwise
2. G is the Murty graph $\mathcal{M}$.

Conjecture 1: An illustration and proof overview

An index-0 operation case in $K_4 \odot K_{3,3}$ leads to $\mathcal{M}$. Here, the addition of a removable edge refers to *index*-0 operation. It can be shown that all other such operations lead to non-Birkohff-von Neumann or non-PM-compact graphs.



$K_4 \odot K_{3,3}$ The Murty graph $\mathcal{M}$

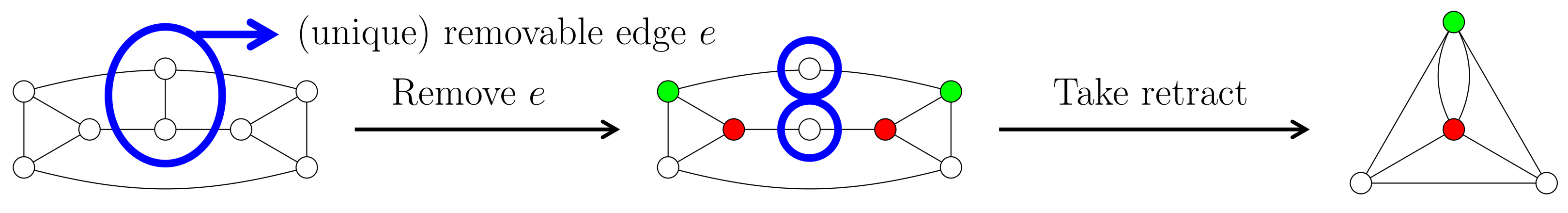
From $K_4 \odot K_{3,3}$ to $\mathcal{M}$ by index-0 operation

Conjecture 2: When all removable edges lead to wheel

Let G be a simple matching covered graph with minimum degree 3 or more. If for each removable edge e, the retract of $G - e$ is either K_4 (with multiple edges such that it does not have a bicycle) or an odd wheel (up to multiple spokes), then G is one of the following:

1. the Bicorn R_8
2. the Tricorn R_{10}
3. the Möius Ladder M_8
4. $K_4 \odot K_{3,3}$
5. an odd wheel

Conjecture 2: An illustration and proof overview

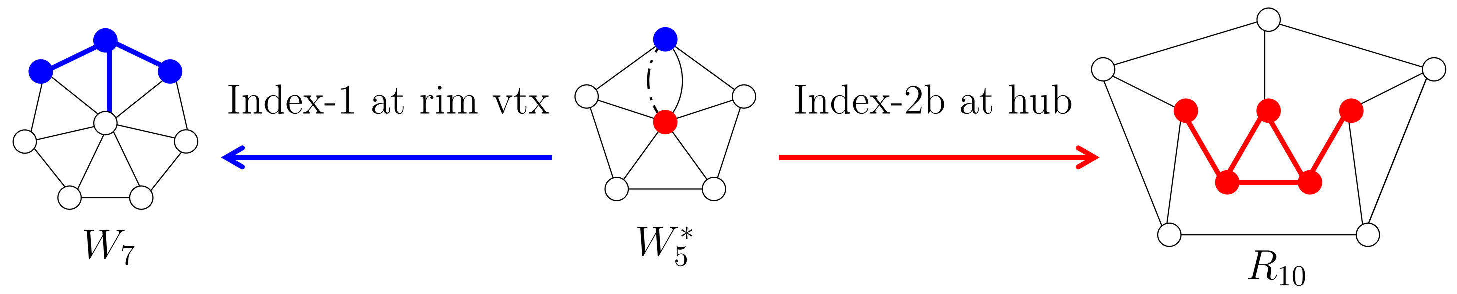


R_8 $R_8 - \{e\}$ K_4

A removable edge in R_8 leading to K_4^*

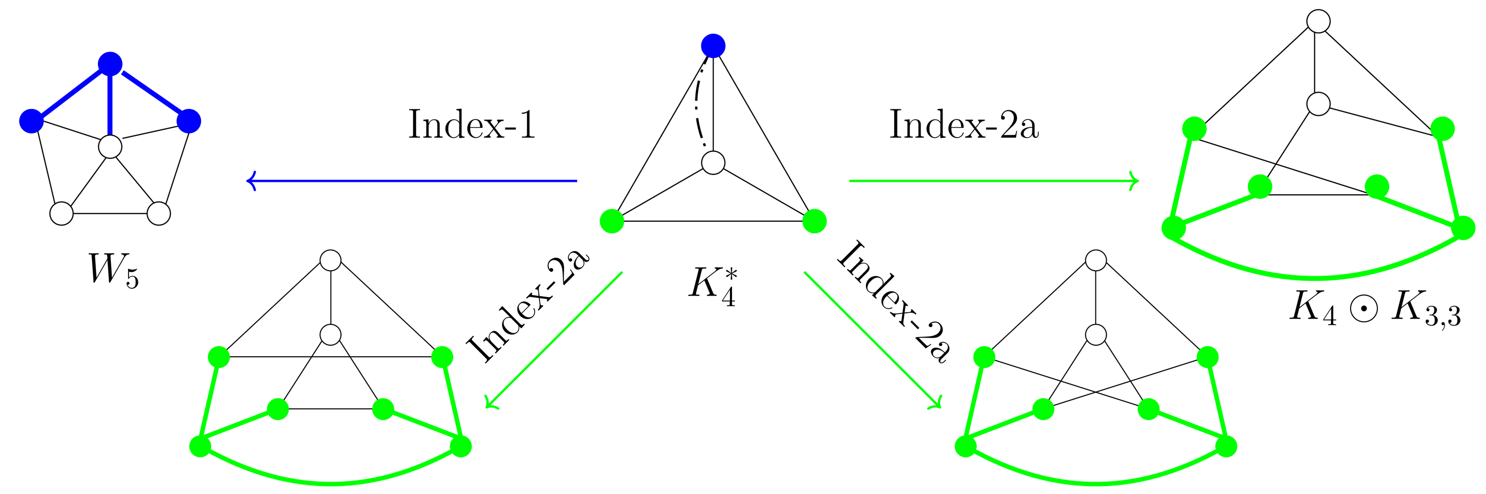
G can be obtained from retract$(G - e)$ either by index-0 or by index-1 or by index-2 operation. These operations have been explained in the diagram.

We conquer the problem by dividing it into two primary cases.



W_7 W_5^* R_{10}

Case 1 : There exists a removable edge e in G such that retract$(G - e)$ is an odd wheel on six or more vertices (up to multiple spokes).



W_5 K_4^* $K_4 \odot K_{3,3}$ R_8 M_8

Case 2 : For each removable edge e in G, the graph retract$(G - e)$ is K_4 (with multiple edges such that it does not have a bicycle).

Who cares... An application

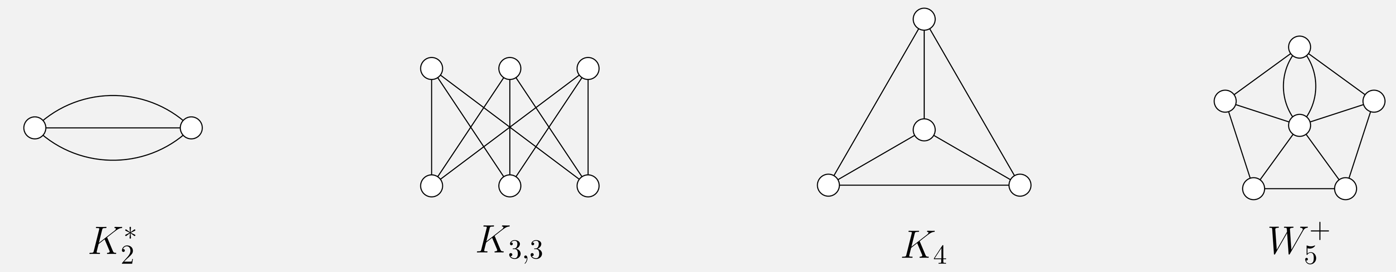
The inspiration for this conjecture arises from Tutte's wheel theorem, highlighting the important role of wheels in several domains of graph theory. Below, an application of this conjecture is presented, offering an alternative proof of the earlier result by Carvalho, Kothari, Wang, and Lin (which uses some advanced theorems in Matching Theory such as brick generation theorem of Norine and Thomas) for characterizing the graphs, that are Birkhoff-von Neumann and PM-compact graph.

Theorem: Birkhoff-von Neumann graphs that are PM-compact

Let G be a matching covered graph with minimum degree 3 or more. The following are equivalent.

1. G is Birkhoff-von Neumann as well as PM-compact.
2. G does not contain a conformal bicycle.
3. G is one of the following:
   a. K_2 (with at least three multiple edges)
   b. $K_{3,3}$
   c. K_4 (up to multiple edges such that it does not have a bicycle)
   d. an odd wheel (up to multiple spokes)
   e. $K_4 \odot K_{3,3}$
   f. the Murty graph (up to multiple edges joining the two noncubic vertices)

The graph gallery



K_2^* $K_{3,3}$ K_4 W_5^+

First four BvN-PMc graphs: a glimpse

What are these creatures: Defining BvN and PMc

In $\mathbb{R}^{|E(G)|}$, perfect matchings of graph G are represented as $0 - 1$ vectors, where 1 indicates an edge's inclusion in the matching. The convex hull of these vectors (i.e., the smallest convex set containing these vectors) is called the *perfect matching polytope* of the graph G (denoted as $\mathcal{PMP}(G)$).

Birkhoff-von Neumann graph : A graph G is *Birkhoff-von Neumann* if $\mathcal{PMP}(G)$ is characterized solely by non-negativity and degree constraints.

Perfect-Matching compact graph : The polytope $\mathcal{P}$ yields the simple undirected graph $\mathbb{S}(\mathcal{P})$ (called the *1-skeleton* of $\mathcal{P}$), where each vertex represents an extreme point of $\mathcal{P}$, and two vertices in $\mathbb{S}(\mathcal{P})$ are adjacent if and only if the corresponding points belong to a common 'edge' (a face of dimension 1) in $\mathcal{P}$. A polytope $\mathcal{P}$ is *compact* if $\mathbb{S}(\mathcal{P})$ is a complete graph. A graph G is *perfect matching-compact* if $\mathcal{PMP}(G)$ is compact. For example, $\overline{C_6}$ (the triangular prism) has 4 perfect matchings and forms a tetrahedron in $\mathbb{R}^9$, with $\mathbb{S}(\mathcal{PMP}(\overline{C_6}))$ as K_4. Thus, $\overline{C_6}$ is PM-compact.

An alternative proof of the above theorem

Let $\mathcal{G}$ denote the class of graphs, that are BvN and PMc. By already established results, '1 $\iff$ 2' holds true. '3 $\implies$ 2' is easily verifiable. We aim to show '2 $\implies$ 3' through induction on the number of edges. From existing theorems, one may deduce that for such a graph G free of conformal bicycle, there exists a removable edge (say e) and the retract of $G - e$ (say J) belongs to $\mathcal{G}$. By the induction hypothesis, J belongs to the aforementioned families. By known results, one need not deal with K_2^* or $K_{3,3}$. Now we have three cases.

Case 1: If G has multiple edges: If its simple graph includes a degree-two vertex, G has an even conformal bicycle. Otherwise, G is K_2^*, K_4^*, an odd wheel (with multiple spokes), or $\mathcal{M}^*$.

Case 2: G has removable edge e, and J is $K_4 \odot K_{3,3}$ or M^* (Invoke Conjecture 1).

Case 3: Neither Case 1 nor Case 2 applies. For each removable edge e, J is K_4 (without a bicycle) or an odd wheel (with multiple spokes). Here, we (invoke Conjecture 2). □

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